

Geodesic Congruences and Their Deformations in Bertrand Space-times

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Abstract

We study the energy conditions and geodesic deformations in Bertrand space-times. We show that these can be thought of as interesting physical space-times in certain regions of the underlying parameter space, where the weak and strong energy conditions hold. We further compute the ESR parameters and analyze them numerically. The focusing of radial time-like and radial null geodesics is shown explicitly, which verifies the Raychaudhuri equation.

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1 Introduction

The Schwarzschild metric, discovered nearly a century ago, remains one of the simplest yet most profound solutions of Einstein’s field equations. One of the reasons for the popularity of the Schwarzschild solution among relativists is that it provides a realistic scenario to describe closed, stable orbits of planets and other heavenly objects. It is however well known that there are other solutions of Einstein’s equations which can also describe such stable, periodic motion. One class of examples was discovered in a remarkable paper by Perlick [1], and these were named “Bertrand space-times” (BSTs). Indeed, Perlick’s classification generalizes the well known Bertrand’s theorem [2] in Newtonian mechanics (an excellent exposition can be found in [3]) to general relativity. The former theorem states that the harmonic oscillator and Kepler potentials are the only spherically symmetric potentials for which bounded orbits are periodic. Based on the standard deductions of the Bertrand’s theorem there was an attempt to generalize its form in the special relativistic case [4]. Perlick’s work determines all static, spherically symmetric space-times in the most general case where one can have stable, closed orbits.

Apart from being interesting from a purely theoretical perspective, BSTs might also be relevant for other reasons. For example, one possibility may be to model a realistic space-time that allows for bounded, periodic orbits. Although such a possibility was ruled out in Perlick’s original work due to the fact that asymptotically flat BSTs (relevant for the motion of objects around an isolated mass) do not seem to satisfy the weak energy condition (WEC) at infinity, asymptotically non-flat BSTs are equally interesting objects, as alternatives to black hole space-times.

This paper studies a class of BSTs from the perspective of the energy conditions and geodesic deformations. We analyze these aspects and find that in certain regions of the parameter space, BSTs do obey the strong and weak energy conditions. We further study the Raychaudhuri equation for BSTs and confirm the geodesic focusing theorem.

The paper is organized as follows. In the next section, we briefly review BSTs and analyze the energy conditions therein. In section 3, we study radial and circular geodesic flows in a class of BSTs and analyze the focusing theorem. Section 4 ends with our conclusions and directions for further study.

2 Energy Conditions in Bertrand space-times

Formally, the definition of a Bertrand space-time [1], [5] arises via a static, spherically symmetric Lorentzian manifold (M, g) whose domain is diffeomorphic to a product manifold $(r_1, r_2) \times S^2 \times \mathbb{R}$ with the metric g given by

$$ds^2 = -e^{2\nu(r)} dt^2 + e^{2\lambda(r)} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad (1)$$

where r ranges in the open interval (r_1, r_2) , θ and ϕ are co-ordinates on the two-sphere. λ and ν are some unspecified functions of r to start with. Such a Lorentzian manifold is called a BST provided there is a circular trajectory passing through each point in the interval (r_1, r_2) , which is stable under small perturbations of the initial conditions.

Starting from this definition, Perlick [1] deduced that there can be two categories of BSTs given by:

$$ds^2 = -\frac{dt^2}{G \mp r^2[1 - Dr^2 \pm \sqrt{(1 - Dr^2)^2 - Kr^4}]^{-1}} + \frac{2[1 - Dr^2 \pm \sqrt{(1 - Dr^2)^2 - Kr^4}]}{\beta^2[(1 - Dr^2)^2 - Kr^4]} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad (2)$$

$$ds^2 = -\frac{dt^2}{G + \sqrt{r^{-2} + K}} + \frac{dr^2}{\beta^2(1 + Kr^2)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad (3)$$

which will be called the Type I and Type II forms of the BST respectively. The parameters D , G and K are real, and β must be a positive rational number.

For mathematical simplicity, we will consider BSTs of type II, defined by the metric of eq.(3). We wish to first understand what type of matter distribution can cause this metric, assuming the Einstein equations to hold. To this end, we construct the Ricci scalar and the energy momentum tensor. The general expressions are too lengthy to reproduce here, and we will frame our arguments based on special cases. Let us begin with the case $K = 0$ for which the type II metric of eq.(3) reduces to

$$ds^2 = -\frac{dt^2}{G + r^{-1}} + \frac{dr^2}{\beta^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad (4)$$

where we will take $G > 0$ to ensure a Lorentzian metric. The Ricci scalar can be

calculated to be ¹

$$R = \frac{4(1+Gr)^2 - \beta^2(4Gr(2+Gr) + 7)}{2r^2(1+Gr)^2} \quad (5)$$

this diverges at $r \rightarrow 0$ and vanishes as $r \rightarrow \infty$. The stress-energy tensor is proportional to

$$T^{\mu\nu} = \text{diag} \left(\frac{(1-\beta^2)(Gr+1)}{r^3}, \frac{\beta^2((Gr+2)\beta^2 - Gr - 1)}{r^2(Gr+1)}, \frac{\beta^2(1-2Gr)}{4r^4(Gr+1)^2}, \frac{\beta^2(1-2Gr)}{4r^4(Gr+1)^2} \right) \quad (6)$$

To analyze the energy conditions, it is convenient to introduce an orthonormal frame that satisfies

$$g_{\mu\nu}e_\alpha^\mu e_\beta^\nu = \eta_{\alpha\beta} \quad (7)$$

where $\eta_{\alpha\beta} = \text{diag}(-1, 1, 1, 1)$ is the flat Lorentzian metric. Since the metric of eq.(4) is diagonal, it is easy to see that a choice of the orthonormal basis is given by $e_\alpha^\mu = \text{diag} \left(\frac{1}{\sqrt{-g_{00}}}, \frac{1}{\sqrt{g_{11}}}, \frac{1}{\sqrt{g_{22}}}, \frac{1}{\sqrt{g_{33}}} \right)$ whence the energy momentum tensor can be written as

$$T^{\mu\nu} = \rho e_0^\mu e_0^\nu + p_1 e_1^\mu e_1^\nu + p_2 e_2^\mu e_2^\nu + p_3 e_3^\mu e_3^\nu \quad (8)$$

and the energy density ρ and the principal pressures p_i , $i = 1 \dots 3$ are

$$\rho = \frac{1-\beta^2}{r^2}, \quad p_1 = \frac{\beta^2(2+Gr) - (1+Gr)}{r^2(1+Gr)}, \quad p_2 = p_3 = \frac{\beta^2(1-2Gr)}{4r^2(1+Gr)^2} \quad (9)$$

It is seen that $\beta > 1$ is ruled out on physical grounds. $\beta = 1$ is somewhat unphysical, as it implies a vanishing energy density in the presence of non zero pressures. For $\beta \rightarrow 1^-$, the weak energy condition [6], [7], $\rho \geq 0$, $\rho + p_i \geq 0$, $i = 1, \dots, 3$ is satisfied for $r < \frac{1}{2G}$. The WEC provides an interesting upper bound on r , and G has to be a small positive number for a physically meaningful solution in this case. For $\beta < 1$, the WEC is satisfied for certain intervals of r , depending on the choice of G . Specifically, it can be checked that for positive values of G (necessary to retain the Lorentzian nature of the metric of eq.(4) at large values of r), the WEC is satisfied for all r .

Let us now turn our attention to non-zero values of K , where the situation is more complicated. With $K \neq 0$, the Ricci scalar diverges at $r \rightarrow 0$, and in the limit $r \rightarrow \infty$, $R_\infty = -6K\beta^2$. The energy density and the principal pressures can

¹Here and in the rest of this section, we set $\theta = \frac{\pi}{2}$ in the final expressions, without loss of generality.

be found by introducing an orthonormal frame analogous to the case $K = 0$, and we find that

$$\rho = \frac{1 - \beta^2 (3Kr^2 + 1)}{r^2} \quad (10)$$

For $r \gg 1$, this implies that the energy density is negative for positive values of K . The situation might be remedied by choosing a negative value of K , but note that this necessitates, from eq.(3) that for $K = -\kappa$ where κ is a positive real number, we require $r < 1/\sqrt{\kappa}$. We can thus choose $\kappa \ll 1$ so that the positivity of the energy density of space-time of eq.(3) is guaranteed for a large range of r . The analysis of the WEC is similar to the case $K = 0$ considered earlier. We will omit the algebraic details here and simply state the result that setting $\beta = 1$ for simplicity, for a given choice of κ (in accordance with the discussion above), the WEC is always satisfied for $r < 1/\sqrt{\kappa}$.

Before we end this section, we will briefly comment on the strong energy condition [6], [7] that follows from eq.(8) : $\rho + \sum_i p_i \geq 0$, $\rho + p_i \geq 0$. We find that for the metric of eq.(3),

$$\rho + \sum_i p_i = \frac{3\beta^2}{2r^2 (Gr + \sqrt{Kr^2 + 1})^2} \quad (11)$$

so that the SEC is satisfied whenever $r < 1/\sqrt{\kappa}$, for positive values of G . This will be important for us in the next section.

To summarize, in this section we have studied the energy conditions of Bertrand space-times of type II, given by the metric of eq.(3). An entirely similar analysis can be carried out for the Type I metric of eq.(2), although the algebraic expressions are complicated. We now proceed to study geodesic flows in BSTs.

3 Geodesic Flows in Bertrand Space-times

The kinematics of geodesic congruence in any space-time can be specified by three quantities: the isotropic expansion, the shear, and the rotation variables. In totality these are generally called the ESR variables, and the evolution of these are guided by the Raychaudhuri equations [8]. Treating the geodesic congruence as a deformable fluid, one can write the evolution equation of the vector between two fluid points. This vector may get deformed as the geodesics flow, and consequently the vector is called the deformation vector. The Raychaudhuri equations connect the evolution of the deformation vector with the curvature of space-time.

In general the deformation vector is called ξ^μ and its rate of change with respect to an affine parameter is given as

$$\dot{\xi}^\mu = B^\mu{}_\nu \xi^\nu, \quad (12)$$

where the affine parameter interval in which the rate is measured is supposed to be small. Here $B^\mu{}_\nu$ is a second rank tensor characterizing the time evolution of the deformation vector,

$$B^\mu{}_\nu = \nabla_\nu u^\mu, \quad (13)$$

where u^μ is a tangent vector field which serves as the first integral of the geodesic equations. Specifically, $u^\nu \nabla_\nu u^\mu = 0$, and choosing a suitable affine parameter one can make $u_\mu u^\mu = -1$ for time-like geodesics while $u_\mu u^\mu = 0$ for a null geodesic. Differentiating the expression in Eq. (12) with respect to the affine parameter one gets

$$\ddot{\xi}^\mu = (\dot{B}^\mu{}_\nu + B^\mu{}_\tau B^\tau{}_\nu) \xi^\nu. \quad (14)$$

The Raychaudhuri equations are obtained by writing $\ddot{\xi}^\mu = -R^\mu{}_{\kappa\tau\nu} u^\kappa u^\nu \xi^\tau$ and equating this with eq(14).

In n space-time dimensions, the general form of the second rank tensor $B_{\mu\nu}$ can be decomposed into irreducible parts as [7]

$$B_{\mu\nu} = \frac{1}{n-1} \Theta h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}, \quad (15)$$

where $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ for u_μ time-like, and Θ is the expansion variable, $\sigma_{\mu\nu}$ is associated with shear and $\omega_{\mu\nu}$ signifies rotation. The physical significance of these variables are nicely explained in [7]. One can explicitly write

$$\Theta = B^\mu{}_\mu, \quad (16)$$

$$\sigma_{\mu\nu} = \frac{1}{2}(B_{\mu\nu} + B_{\nu\mu}) - \frac{1}{n-1} \Theta h_{\mu\nu}, \quad (17)$$

$$\omega_{\mu\nu} = \frac{1}{2}(B_{\mu\nu} - B_{\nu\mu}). \quad (18)$$

and the ESR variables are generally denoted by Θ , σ^2 and ω^2 . From Eq. (13) it is seen that if one knows the form of u^μ one can calculate $B_{\mu\nu}$, and hence the ESR parameters. These are expected to give us information about geodesic flows and their properties in BSTs.

3.1 Geodesics in BSTs of Type II : General Considerations

We now focus on Type II BSTs, and further simplify the situation by choosing $\theta = \pi/2$, so that we are on the equatorial plane. In that case we have the metric

$$ds^2 = -\frac{dt^2}{G + \sqrt{K + r^{-2}}} + \frac{dr^2}{\beta^2(1 + Kr^2)} + r^2 d\phi^2. \quad (19)$$

The geodesic equations can now be written down. The first one is obvious from the form of the above line element,

$$\frac{\dot{t}}{G + \sqrt{K + r^{-2}}} = C, \quad (20)$$

where C is a constant of integration. This equation can also be written as

$$\ddot{t} + \frac{\dot{t}\dot{r}}{r^2\sqrt{1 + Kr^2}(G + \sqrt{K + r^{-2}})} = 0. \quad (21)$$

The other geodesic equations are:

$$\ddot{r} + \frac{\beta^2\sqrt{1 + Kr^2}\dot{t}^2}{2r^2(G + \sqrt{K + r^{-2}})^2} - \frac{Kr\dot{r}^2}{1 + Kr^2} - r\beta^2(1 + Kr^2)\dot{\phi}^2 = 0, \quad (22)$$

and

$$\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} = 0. \quad (23)$$

On a radial geodesic Eq. (20) holds and more over $\dot{\phi} = 0$. For a time-like geodesic ($u^\mu u_\mu = -1$), one can write

$$-\frac{\dot{t}^2}{G + \sqrt{K + r^{-2}}} + \frac{\dot{r}^2}{\beta^2(1 + Kr^2)} + r^2\dot{\phi}^2 = -1. \quad (24)$$

Using the above equation one can calculate the value of $u^r = \dot{r} = dr/d\lambda$ on a radial geodesic. Here λ is an affine parameter. The value of u^r comes out as

$$u^r = \frac{dr}{d\lambda} = \beta\sqrt{(1 + Kr^2)\left[C^2\left(\sqrt{K + r^{-2}} + G\right) - 1\right]} \quad (25)$$

The above equation is for the outgoing radial geodesic directed away from the origin. Note that, assuming $Kr^2 + 1 > 0$ (see discussion in section 2), this implies that there is a turning point of the outgoing radial time-like geodesics,

for $C^2 (\sqrt{K + r^{-2}} + G) = 1$. This implies that there is a maximum value of r at which outgoing radial geodesics stop. This is analogous to the case of the Schwarzschild black hole, where it is known that such turning points occur for non-marginally bound radial time-like geodesics. The other components of the tangent vector u^μ on the radial geodesic are

$$u^t = \frac{dt}{d\lambda} = C \left(G + \sqrt{K + r^{-2}} \right), \quad u^\phi = \frac{d\phi}{d\lambda} = 0. \quad (26)$$

Having calculated the relevant components of the tangent vectors u^μ on a radial time-like geodesic of Type II BSTs, one can compute the components of the B^μ_ν tensor for radial time-like geodesics.

For future reference, let us also list the components of the tangent vector for the radial null and the circular time-like geodesics. For the former, we obtain

$$\begin{aligned} u^t &= C \left(G + \sqrt{K + r^{-2}} \right), \\ u^r &= \beta C \sqrt{(Kr^2 + 1) \left(G + \sqrt{K + r^{-2}} \right)} \\ u^\phi &= 0, \end{aligned} \quad (27)$$

where C is defined in Eq. (20). It is interesting to note that for null radial geodesic case there is no turning point (for $G > 0$ as is always assumed in this article) as was present for the time-like radial geodesics. This implies that light propagating away along the radial direction in BST of type II is not bound to return after travelling a finite distance. In the later part of this article we will see that although outgoing null radial geodesics do not have a turning point, an outgoing radial null geodesic congruence does focus away from the origin.

For the circular time-like geodesics, we find

$$\begin{aligned} u^t &= \sqrt{2}r \sqrt{\frac{\sqrt{K + r^{-2}} \left(G + \sqrt{K + r^{-2}} \right)^2}{2r \left(G\sqrt{Kr^2 + 1} + Kr \right) + 1}} \\ u^r &= 0, \\ u^\phi &= \sqrt{\frac{1}{r^2 \left[2r \left(G\sqrt{Kr^2 + 1} + Kr \right) + 1 \right]}} \end{aligned} \quad (28)$$

From the above expressions of the tangent vectors one can see that there exists an upper bound on the radial distance up to which BST of type II can accommodate time-like circular geodesics. The upper limit is given by the inequality

$$2r \left(G\sqrt{Kr^2 + 1} + Kr \right) + 1 > 0. \quad (29)$$

If $K = 0$ the above inequality is satisfied for all r . On the other hand if $G = 0$ and $K = -\kappa$ where $\kappa > 0$, the upper bound is given by $1/\sqrt{2\kappa}$. In general when $G > 0$ and $K < 0$ the upper bound on r has to be evaluated by solving the inequality in eq.(29). If both G and K are greater than zero the inequality in eq.(29) is trivially satisfied but as we have seen in section 2 that in this case the WEC and SEC are violated for $r \gg 1$.

To summarize, in this subsection, we have considered the BST of type II (eq.(3)), and calculated the first integrals of the geodesic equation for radial time-like, radial null and circular time-like vectors. These can be used in a straightforward manner to evaluate the ESR parameters for BSTs, which we now turn to.

3.2 The ESR variables for Type II BSTs

In this subsection, we compute the ESR parameter for BSTs of type II. Consider first the radial time-like geodesics. We start from eq.(13), from which we can write

$$B^\mu{}_\nu = \frac{\partial u^\mu}{\partial x^\nu} + \Gamma^\mu_{\nu\rho} u^\rho, \quad (30)$$

The non-zero components of $B^\mu{}_\nu$, required to evaluate the ESR variables for the radial time-like geodesic flow, in the equatorial plane for the Type II Bertrand space-time, are listed below:

$$\begin{aligned} B^t{}_t &= \frac{\beta \sqrt{C^2(G + \sqrt{K + r^{-2}}) - 1}}{2r^2(G + \sqrt{K + r^{-2}})}, \quad B^t{}_r = \frac{C}{2r^2 \sqrt{1 + Kr^2}} \\ B^r{}_t &= -\frac{\beta^2 C \sqrt{1 + Kr^2}}{2r^2(G + \sqrt{K + r^{-2}})}, \quad B^r{}_r = -\frac{\beta C^2}{2r^2 \sqrt{C^2(G + \sqrt{K + r^{-2}}) - 1}} \\ B^\phi{}_\phi &= \frac{\beta}{r} \sqrt{(1 + Kr^2)[C^2(G + \sqrt{K + r^{-2}}) - 1]}. \end{aligned} \quad (31)$$

Now from eq.(16), we get

$$\Theta = \frac{\beta \sqrt{1 + Kr^2} (Gr + \sqrt{1 + Kr^2}) \left[2C^2 - \frac{r(3 + 2Kr^2 + 2Gr\sqrt{1 + Kr^2})}{\sqrt{1 + Kr^2}(Gr + \sqrt{1 + Kr^2})^2} \right]}{2r^2 \sqrt{C^2(G + \sqrt{K + r^{-2}}) - 1}}. \quad (32)$$

The shear coefficient squared, $\sigma^2 \equiv \sigma_{\mu\nu} \sigma^{\mu\nu}$, for the radial time-like geodesics comes out as

$$\sigma^2 = \frac{\beta \mathcal{P}(r)}{\mathcal{Q}(r)}, \quad (33)$$

where $\mathcal{P}(r)$ and $\mathcal{Q}(r)$ are functions of r , given by

$$\begin{aligned}\mathcal{P}(r) &= 2Kr^3(2C^2G - 1) + 2r^2\sqrt{Kr^2 + 1}(G(C^2G - 1) + C^2K) \\ &\quad + r(4C^2G - 3) + 2C^2\sqrt{Kr^2 + 1} \\ \mathcal{Q}(r) &= 2r^2\left(Gr + \sqrt{Kr^2 + 1}\right)\sqrt{C^2\left(G + \frac{\sqrt{Kr^2 + 1}}{r}\right) - 1}\end{aligned}\quad (34)$$

The rotation parameter for the radial time-like geodesics, $\omega^2 \equiv \omega_{\mu\nu}\omega^{\mu\nu} = 0$. Before we move on, let us make a few comments. First of all, note that Θ and σ^2 diverge at $r = 0$, corresponding to the singularity of the BST at that point. These also diverges at $C^2(\sqrt{K + r^{-2}} + G) = 1$, the turning point for outgoing radial time-like geodesics (see discussion after eq.(25)) and indicates that the geodesics focus or de-focus at the turning point.

Next we present the ESR parameters for the circular time-like geodesics. For these, the B^μ_ν components are given as

$$\begin{aligned}B^t_r &= -\frac{(G + 2GKr^2 + 2Kr\sqrt{1 + Kr^2})}{\sqrt{2r}(1 + Kr^2)^{3/4}(1 + 2Kr^2 + 2Gr\sqrt{1 + Kr^2})^{3/2}} \\ B^r_t &= \frac{\beta^2(1 + Kr^2)^{3/4}}{\sqrt{2r}(Gr + \sqrt{1 + Kr^2})(1 + 2Kr^2 + 2Gr\sqrt{1 + Kr^2})^{1/2}} \\ B^r_\phi &= -\frac{\beta^2(1 + Kr^2)^{3/4}}{(1 + 2Kr^2 + 2Gr\sqrt{1 + Kr^2})^{1/2}}, \\ B^\phi_r &= -\frac{(G + 2GKr^2 + 2Kr\sqrt{1 + Kr^2})}{r\sqrt{1 + Kr^2}(1 + 2Kr^2 + 2Gr\sqrt{1 + Kr^2})^{3/2}}.\end{aligned}\quad (35)$$

and the diagonal elements of B^μ_ν are all zero. This implies the expansion coefficient $\Theta = 0$ for the circular time-like geodesics. The shear coefficient is

$$\sigma^2 = \frac{\beta^2[\sqrt{1 + Kr^2}(1 + 4Kr^2) + Gr(3 + 4Kr^2)]^2}{4r^2\sqrt{1 + Kr^2}(Gr + \sqrt{1 + Kr^2})(1 + 2Kr^2 + 2Gr\sqrt{1 + Kr^2})^2}.\quad (36)$$

The rotation parameter ω^2 for the circular time-like geodesics is

$$\omega^2 = \frac{\beta^2(Gr + \sqrt{1 + Kr^2})}{4r^2\sqrt{1 + Kr^2}(1 + 2Kr^2 + 2Gr\sqrt{1 + Kr^2})^2}.\quad (37)$$

To discuss the behavior of these parameters, let us first consider the case $K = 0$. In this case, the shear and rotation parameters reduce to

$$\begin{aligned}\sigma^2 &= \frac{\beta^2(3Gr + 1)^2}{4r^2(Gr + 1)(2Gr + 1)^2} \\ \omega^2 &= \frac{\beta^2(Gr + 1)}{4r^2(2Gr + 1)^2}\end{aligned}\quad (38)$$

These are positive everywhere, for any value of β and go to zero in the limit of infinite r (remember that G is positive). For $G > 0$ and $K < 0$ the analysis of the shear and the rotation parameter cannot be simply guessed by the expressions in eqs.(36) and (37) as in this case one can have an upper bound of the radial distance up to which circular time-like geodesics can be obtained in Type II BST.

Finally, let us briefly discuss null geodesics. For the radial null geodesics the evolution tensor $\tilde{B}_{\mu\nu}$ is defined as

$$\tilde{B}_{\mu\nu} = P^\lambda_\mu B_{\lambda\kappa} P^\kappa_\nu, \quad (39)$$

where the projection tensor $P_{\alpha\beta}$ is

$$P_{\alpha\beta} = g_{\alpha\beta} + n_\alpha u_\beta + n_\beta u_\alpha. \quad (40)$$

Here u_α is the first integral of null geodesic equations i.e., $u^\beta \nabla_\beta u_\alpha = 0$ and $u_\alpha u^\alpha = 0$. The vector n_α satisfies the following conditions:

$$n_\alpha n^\alpha = 0, \quad n_\alpha u^\alpha = -1, \quad u^\beta \nabla_\beta n_\alpha = 0. \quad (41)$$

Now the evolution tensor (in an effectively one dimensional space) becomes

$$\tilde{B}_{\mu\nu} = \Theta P_{\mu\nu} + \tilde{\sigma}_{\mu\nu} + \tilde{\omega}_{\mu\nu}, \quad (42)$$

where

$$\begin{aligned} \Theta &= \tilde{B}^\mu_\mu = B^\mu_\mu, \\ \tilde{\sigma}_{\mu\nu} &= \frac{1}{2}(\tilde{B}_{\mu\nu} + \tilde{B}_{\nu\mu}) - \Theta P_{\mu\nu}, \\ \tilde{\omega}_{\mu\nu} &= \frac{1}{2}(\tilde{B}_{\mu\nu} - \tilde{B}_{\nu\mu}). \end{aligned} \quad (43)$$

The vector n^α is tangent to the inward radial geodesics. If we take

$$\begin{aligned} n^t &= -\frac{1}{2C}, \\ n^r &= -\frac{\sqrt{r}\beta C \sqrt{(Kr^2 + 1)(Gr + \sqrt{Kr^2 + 1})}}{2C^2 (Gr + \sqrt{Kr^2 + 1})}, \\ n^\phi &= 0, \end{aligned} \quad (44)$$

then these components n^α satisfy all the conditions in Eq. (41). Using these we can find all the components of P^λ_μ . The only non zero component of P^λ_μ turns out to be P^ϕ_ϕ which is given as

$$P^\phi_\phi = 1. \quad (45)$$

Using the above component of the projection tensor one finds that there is only one non-zero element of \tilde{B}^μ_μ , namely, \tilde{B}^ϕ_ϕ . Consequently, for the radial null geodesics, we obtain

$$\Theta = \tilde{B}^\phi_\phi = \beta C r^{-\frac{3}{2}} \sqrt{(K r^2 + 1) \left(G r + \sqrt{K r^2 + 1} \right)} \quad (46)$$

The above expression of the expansion variable for the null radial geodesic congruence for Type II BST shows that Θ has a singularity at $r = 0$ where BST of Type II itself is singular. Unlike the outgoing time-like radial geodesics the expansion variable for the outgoing null radial geodesics do not have any other singularity.

3.3 Analysis of BST Type II spacetime in terms of the ESR parameters

Having obtained the ESR parameters for BSTs of type II, let us analyze them in some detail. First, we focus on radial time-like geodesics. In this case, the ESR parameters can be obtained by setting $K = 0$ in eq.(32) and in eqs.(33), (34). As mentioned in the discussion after eq.(6), we can consider two cases here, namely $\beta = 1$ and $\beta < 1$. Let us focus on the former case. Here, the expression for the expansion parameters assumes a simple form

$$\Theta = \frac{(Gr + 1) \left(2C^2 - \frac{r(2Gr+3)}{(Gr+1)^2} \right)}{2r^2 \sqrt{C^2 \left(G + \frac{1}{r} \right) - 1}} \quad (47)$$

with σ^2 given from eqs.(33) and (34), with $K = 0$.

We wish to understand the behavior of Θ and σ as a function of the affine parameter, λ . (Note, however that in our method, no initial condition on the parameter Θ or σ is possible, unlike the case where one integrates the full set of Raychaudhuri equations with given initial conditions [9], [10] (see also [11]). We will proceed, keeping this in mind). To this end, we first numerically solve eq.(25) (with chosen upper and lower limits of the affine parameter) to express the radial coordinate r as a function of λ . This is then fed back in eq.(47) to obtain the behavior of θ as a function of λ . For illustration purpose, we have chosen $G = 10^{-3}$, and set $C = 1$, and the lower and upper limits of the affine parameter λ have been set to -0.2 and 1.5 . The upper limit of λ is chosen so that r varies from zero to the turning point of u^r , which can be seen to be $r \sim 1$ in this case

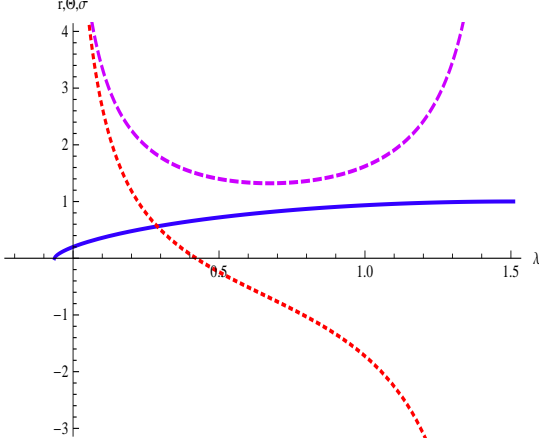


Figure 1: Numerical solutions for r (solid blue), Θ (dotted red), and $\sigma \equiv \sqrt{\sigma_{\mu\nu}\sigma^{\mu\nu}}$ (dashed magenta) for radial time-like geodesics in type II BST, as a function of λ for $K = 0$. For details, see text.

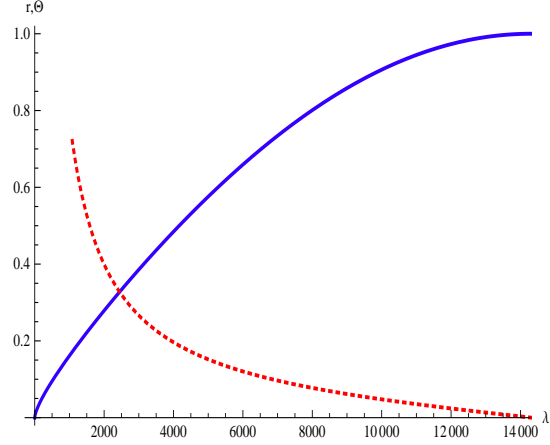


Figure 2: Numerical solutions for r (solid blue) and Θ (dotted red) for radial null geodesics in type II BST, as a function of λ for the value $K = -10^{-6}$. For details, see text.

(these numbers are simply for illustration). The result is shown in fig.(1), where the solid blue, dotted red and the dashed magenta curves correspond to numerical solutions for r , Θ and σ respectively, as a function of the affine parameter, with the chosen initial condition. From the figure, we see that $\frac{d\theta}{d\lambda}$ is always negative, confirming the focusing theorem [6], [7] given the validity of the SEC of eq.(11). Also, Θ diverges at the upper and lower limits of r , signaling the turning points of the geodesics. We find that the case $K \neq 0$ for radial time-like geodesics follow the same qualitative behavior.

To illustrate the case of null geodesics, we have taken $K \neq 0$. We have followed a numerical procedure similar to that alluded to above, and chosen $K = -10^{-6}$, $G = 10^{-2}$, and C and β has been set to unity. Here, the lower and upper limits of the affine parameter has been set to -0.2 and 1.389 respectively. Using the same numerical procedure as above, we solve for the expansion parameter of eq.(46), and this is illustrated in fig.(2), where we have multiplied Θ by a factor of 10^3 to display the curves on the same graph. Note that in this case the expansion parameter diverges at $r = 0$ as expected from eq.(46).

Before we conclude, we briefly mention the expansion parameter for the BST of type II in the four dimensional case. The details are unimportant here, and these can be worked out exactly like the case $\theta = \frac{\pi}{2}$. For radial time-like geodesics,

the expansion parameter is $\Theta = \beta \mathcal{A}(r)/\mathcal{B}(r)$ where

$$\begin{aligned}\mathcal{A}(r) &= (4r^2\sqrt{Kr^2+1}(C^2G^2+C^2K-G) + 4Kr^3(2C^2G-1) \\ &\quad + r(8C^2G-5) + 4C^2\sqrt{Kr^2+1}) \\ \mathcal{B}(r) &= 2r^2\left(Gr + \sqrt{Kr^2+1}\right)\sqrt{C^2\left(G + \frac{\sqrt{Kr^2+1}}{r}\right) - 1}\end{aligned}\quad (48)$$

For radial null geodesics in four dimensions, the expansion parameter turns out to be twice that of eq.(46). The shear parameter for the radial time-like case yields a lengthy expression analogous to eqs.(33) and (34), which we omit for brevity, and is zero for the radial null case, as before.

4 Conclusions and Discussions

In this paper, we have considered the energy conditions and geodesic deformations in Bertrand space-times. For simplicity, we have chosen the BST of type II (eq.(3)), although we believe that the qualitative results will remain unchanged even for a type I BST. As the metric of BST of type II contains three parameters we had to check the probable ranges of these parameters which gives the spacetime physical significance. We have explicitly checked the weak and strong energy conditions for type II BSTs, and verified their validity within certain ranges of parameters. The ESR parameters for both time-like and null geodesics in the BST type II spacetime are calculated as functions of the radial coordinate. For simplicity most of the calculations are done in the equatorial plane. We have not explicitly solved the Raychaudhuri equation in the above mentioned spacetime but indirectly obtained its solution by expressing the radial coordinate in terms of the affine parameter in the ESR variables. In such a situation the ESR variables becomes functions of the affine parameter and these variables are now valid solutions of the Raychaudhuri equation for geodesic deformations in type II BSTs. While analyzing the solutions of the Raychaudhuri equation for geodesic deformations in type II BSTs we have confirmed the focusing theorem numerically.

The behavior of the solution of the Raychaudhuri equation for geodesic deformations in type II BST for the time-like geodesic congruence is analyzed in this article with some suitable choice of parameter values K , G and β . In the specific case chosen it is seen that generally the outgoing radial time-like/null

geodesics diverge from $r = 0$, which is a singular point in this spacetime where the Ricci scalar diverges. The outgoing radial time-like geodesics do not diverge indefinitely, they do converge again at some other value of r which turns out to be a turning point in type II BST. Whereas an outgoing radial null geodesic congruence do not converge at any finite r but it does show focusing property. At the turning point for the time-like radial geodesics, spacetime is not singular but the radial component of the tangent vector to the radial geodesics vanish at this point. For the radial geodesics the rotation parameter is always zero but the time-like radial geodesics do show extreme shear near $r = 0$ and the turning point. The circular time-like geodesics do not show any focusing behavior, as expected. But the circular orbits do show rotation.

Our analysis points to the fact that BSTs can be thought of as interesting realistic examples of static, spherically symmetric space-times, which are asymptotically non-flat, and allow for stable, periodic orbits. This might be significant in astrophysical scenarios : for example, one might ask if a realistic space-time near a compact object can be modeled via BST of type II or I. In this article the main attention was given to the geodesics of BST of type II and its ESR variables to understand the effect of spacetime curvature and probable singularities. To properly utilize BSTs, one must also have to think of the source of such kind of spacetime's and in future works one needs to look at this important aspect. For completeness, it will be of interest to study, in the same manner, BSTs of type I and analyze the parameter space of this theory. We leave this for a future publication.

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